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Scalar scattering via conformal higher spin exchange

Euihun Joung,[a,b] **Simon Nakach**[c] **and Arkady A. Tseytlin**[c]

[a] *School of Physics and Astronomy, Seoul National University, Seoul 151-747, Korea*

[b] *Gauge, Gravity & Strings, Center for Theoretical Physics of the Universe, Institute for Basic Sciences, Daejeon 34047, Korea*

[c] *Theoretical physics group, Blackett Laboratory, Imperial College London, SW7 2AZ, U.K.*

E-mail: euihun.joung@snu.ac.kr, nakach@ic.ac.uk, tseytlin@ic.ac.uk

Abstract: Theories containing infinite number of higher spin fields require a particular definition of summation over spins consistent with their underlying symmetries. We consider a model of massless scalars interacting (via bilinear conserved currents) with conformal higher spin fields in flat space. We compute the tree-level four-scalar scattering amplitude using a natural prescription for summation over an infinite set of conformal higher spin exchanges and find that it vanishes. Independently, we show that the vanishing of the scalar scattering amplitude is, in fact, implied by the global conformal higher spin symmetry of this model. We also discuss one-loop corrections to the four-scalar scattering amplitude.

Keywords: Higher Spin Gravity, Higher Spin Symmetry, Scattering Amplitudes, Anomalies in Field and String Theories

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Contents

1 Introduction

Higher spin theories containing infinite number of particles pose a challenge of how to define them at the quantum level in a way consistent with their large amount of symmetry. One particular issue is how to treat sums over infinite number of spins. This question was recently addressed on examples of simplest higher spin partition functions in [1] following [2–8].

Our aim will be to study this issue in the context of S-matrix of scalars interacting via exchange of an infinite set of higher spin fields. This is an analog of the Veneziano amplitude in string theory where the infinite tower of exchanged fields are massive. This set-up was originally discussed in [9] where a tree-level scalar scattering amplitude with standard massless higher spin particles exchange was considered. Since an interacting theory of massless higher spin particles ought to be not well-defined in flat space (cf. [10–12]) the computation of [9] is, however, hard to embed into a consistent theory.

Here instead we shall consider a model where the scalars interact through exchange of a tower of *conformal higher spin* fields. Conformal Higher Spin (CHS) theories are generalisations of $d = 4$ Maxwell ($s = 1$) and Weyl ($s = 2$) theories that describe pure spin s states off shell, i.e. have maximal gauge symmetry consistent with locality at the expense of having higher-derivative kinetic terms [13] (see also [7, 14–17]). In contrast to the two-derivative massless higher spin theory, the CHS theory (that can be defined at the full non-linear level as the UV singular local part of the induced action of free scalars with higher spin background fields coupled to all conserved spin s scalar currents [15–17]) may be viewed as a formally consistent (but a priori non-unitary) interacting gauge theory when expanded near flat space.

To introduce a particular model which we shall study in this paper, let us first recall the basics of vectorial AdS/CFT duality (see, e.g., [6, 8, 18]). Consider a free CFT$_d$ of N complex scalar fields

$$S = \int d^d x \ \vec{\chi}^* \cdot \partial^2 \vec{\chi} \,, \tag{1.1}$$

with primary conformal operators being on-shell-conserved traceless currents $J_{\mu_1 \dots \mu_s}$ of dimension $\Delta = d - 2 + s$. The latter are bilinear $\Upsilon(N)$ singlets (see [19])

$$J_s(\vec{\chi}) = \vec{\chi}^* \cdot \mathcal{J}_s \, \vec{\chi} \sim \vec{\chi}^* \cdot \partial^s \, \vec{\chi} \,, \qquad\qquad s = 0, 1, 2, \dots \,, \tag{1.2}$$

where $\mathcal{J}_s$ is an appropriate differential operator. Introducing source fields $h_s(x)$ for all J_s and integrating out $\vec{\chi}$, one gets a generating functional for connected correlators of all currents

$$\Gamma[h] = N \, \log \det \Big(- \partial^2 + \sum_s h_s \, \mathcal{J}_s \Big) \,. \tag{1.3}$$

The d-dimensional fields h_s may be viewed as gauge fields for the symmetries of the free classical scalar theory with linearised differential and algebraic ("trace shifting") symmetries generalising the reparametrization and Weyl symmetry of the Weyl gravity. They can thus be identified with the CHS fields.[1]

The same functional $\Gamma[h]$ (1.3) should follow from the Vasiliev's massless higher spin theory [20–22] in AdS$_{d+1}$ upon integrating over the AdS$_{d+1}$ Fronsdal fields Φ_s with Dirichlet boundary conditions ($\Phi_s\big|_{\partial \mathrm{AdS}} = h_s$). The number of scalars N then plays the role of the inverse coupling of the higher spin theory in AdS$_{d+1}$ (appearing in front of its classical action). All quantum (order $N^0, N^{-1}, \dots$) corrections to the generating functional computed from the Vasiliev's theory should then vanish to match the boundary theory result.[2]

[1]Demanding invariance under non-linear symmetries for a particular subset of fields may require introducing extra terms non-linear in h_s (like in scalar electrodynamics or in covariant coupling to a curved metric). However, being local (involving powers of h_s fields at the same point), they would not change the values of the CFT correlators of primary operators J_s at separated points.

[2]More precisely, what should vanish are corrections to derivatives of the generating functional at separated points.

The quadratic term in h_s term of $\Gamma[h]$ in (1.3) is

$$\Gamma_2[h] = N \sum_s \int h_s K_s h_s \,, \tag{1.4}$$

$$\tag{1.5}$$

with

$$K_s \sim N^{-1} \langle J_s(x) J_s(x') \rangle \sim \mathrm{P}_s \, |x-x'|^{4-2d-2s} \sim \mathrm{P}_s \, \partial^{2s+d-4} \, \delta^{(d)}(x-x') \, \log \Lambda + \dots \tag{1.6}$$

where P_s is the transverse traceless projector and Λ is a UV cutoff. From now on, we assume d is even. Thus the UV singular part of Γ_2 is proportional to the collection of CHS kinetic terms $\int d^d x \ h_s \, \mathrm{P}_s \, \partial^{2s+d-4} \, h_s$.

Suppose now we start with $N+1$ scalar fields, $\vec{\chi}$ and ϕ, couple them to the CHS fields h_s via the currents $J_s(\vec{\chi})+J_s(\phi)$ and integrate out only N scalars $\vec{\chi}$. The resulting effective theory will contain the remaining scalar ϕ coupled to the CHS fields h_s described by the induced action, i.e.

$$S[\phi, h] = \int d^d x \Big[\phi^* \partial^2 \phi + \sum_s h_s \, J_s(\phi) \Big] + \Gamma[h] \,, \tag{1.7}$$

where $\Gamma[h] = N \sum_s [\int h_s K_s h_s + \mathcal{O}(h^3)]$. The UV singular local part of $\Gamma[h]$ may be identified with a non-linear CHS action [15–17]. One may then compute the S-matrix for ϕ due to the exchange of the tower of all CHS fields h_s. Assuming N (or the inverse CHS theory coupling) is large we may treat self-interactions of h_s in perturbation theory.

While a non-trivial S-matrix for ϕ is not a natural observable in the boundary CFT_d (which is a free theory from the start) this set-up is in a sense a higher spin theory analog of the computation of the 4d gluon S-matrix from the AdS_5 point of view [23] where one first "integrates out" $\mathrm{SU}(N)$ gauge vectors to "build" the bulk geometry, and then considers the scattering of extra gluons on a probe 3-brane.

In general, one may study the case when the CHS part $\Gamma[h]$ of the model (1.7) is given by either the full non-local induced action (i.e. with kinetic term $\mathrm{P}_s \, \partial^{2s+d-4} \log(\partial^2/\Lambda^2)$) or simply its local UV singular part $\mathrm{P}_s \, \partial^{2s+d-4} \, \log \Lambda$. The latter choice is preferable when trying to include also self-interactions of h_s: the finite part of the full induced action is a priori anomalous, breaking the classical algebraic symmetries of the CHS fields.[3] At the same time, the local $\log \Lambda$ part of $\Gamma[h]$ is invariant under the symmetries of the CHS theory [16, 17].

In what follows we shall study the model (1.7) viewed as a *local* CHS theory interacting with a free conformal scalar matter, i.e. assume that only the local part of $\Gamma[h]$ defining the

[3]The anomalous part of the effective action does not, however, contribute to the correlation functions of conformal current operators at separated points (the anomaly expressions contain at least two fields at the same point). For example, a scalar ϕ coupled to the background metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in a reparametrisation and Weyl-covariant way (i.e. with $\frac{(d-2)}{4(d-1)} \, R \, \phi^* \phi$ term included) has Weyl-anomalous (starting with cubic $(h_2)^3$ order) effective action but its UV divergent part $\sim$ (Weyl tensor)^2 is Weyl-invariant to all orders.

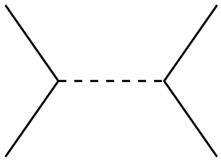

Figure 1. Tree-level diagram.

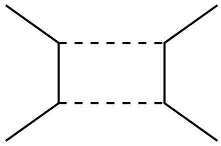

Figure 2. Box diagram.

CHS action $S[h]$ is kept with coefficient $\kappa \sim N$ as the (inverse) coupling constant. Starting with (1.7) and rescaling h_s as $h_s \to \sqrt{N}\, h_s$, we get

$$S[\phi, h] = \int d^d x \left[\phi^* \, \partial^2 \, \phi + \sum_s h_s \, \mathrm{P}_s \, \partial^{2s+d-4} \, h_s + \frac{1}{\sqrt{\kappa}} \left(\sum_s h_s \, \phi^* \mathcal{J}_s \, \phi + h^3 \right) + \mathcal{O}\left(\frac{1}{\kappa} \, h^4 \right) \right]. \tag{1.8}$$

Thus at the leading $\frac{1}{\kappa}$ order we get the four-scalar tree level diagram (figure 1) with two $\frac{1}{\sqrt{\kappa}}$ vertices. Here the solid line (—) stands for the scalar ϕ propagator and the dashed line (- - -) for all CHS propagators. We shall explicitly compute the corresponding amplitude below.

In addition, we shall also discuss the one-loop corrections to 4-scalar scattering. An example of such one-loop order $\frac{1}{\kappa^2}$ diagram is the 1-PI one (figure 2) with four $\frac{1}{\sqrt{\kappa}}$ vertices.

The one-loop four-scalar amplitude of order $\frac{1}{\kappa^2}$ receives also contributions from non-1-PI diagrams which are the tree-diagrams in figure 1 where the scalar legs, the CHS propagators and the vertices get the $\frac{1}{\kappa}$ corrections due to the scalar self-energy diagram, the CHS self-energy diagram, and the charge-renormalization diagram.

We will start in section 2 with a description of the model of a free scalar field coupled to a tower of CHS fields. In section 3 we will compute the tree level amplitude corresponding to figure 1 using a particular regularisation prescription for the sum over all spins. The resulting amplitude will have a special scale-invariant form and will vanish due to the constraints of the massless scalar kinematics.

As we shall show in section 4 the vanishing of the four-scalar amplitude is, in fact, implied by the global CHS symmetry of the model. This will thus justify our choice of the summation over spins prescription.

In section 5 we will consider the one-loop amplitude given by figure 2 and similar diagrams limiting the computation to the local UV divergent $(\phi^*\phi)^2$ contribution to it. Some concluding remarks will be made in section 6.

In appendix A we will review the global CHS symmetry transformations. In appendix B we will present the explicit form of the cubic and quartic vertices in the CHS action relevant for the computations in section 5. The transverse traceless gauge fixing and the corresponding ghost action will be discussed in appendix C.

2 Scalar field interacting with conformal higher spin fields

Let us start with a free complex massless scalar ϕ with the flat space action

$$S_{\text{free}}[\phi] = \int d^d x\, \phi^* \,\partial^2\, \phi\,. \tag{2.1}$$

This free theory admits infinitely many conserved (on-shell) currents, which are traceless due to conformal invariance. A generating function for such traceless conserved currents may be defined using an auxiliary vector u_μ as (see [19])

$$J(x,u) = \sum_{s=0}^{\infty} \frac{1}{s!}\, J^{\mu_1\cdots\mu_s}(x)\, u_{\mu_1}\cdots u_{\mu_s}\ . \tag{2.2}$$

Here

$$J(x,u) = \Pi_d(u,\partial_x)\, \mathfrak{J}(x,u)\,, \tag{2.3}$$

where $\mathfrak{J}(x,u)$ is the generating function of traceful currents

$$\mathfrak{J}(x,u) = \phi^*\Big(x + \frac{i}{2}\,u\Big)\,\phi\Big(x - \frac{i}{2}\,u\Big)\,, \tag{2.4}$$

and Π_d is an operator mapping the traceful currents into traceless currents [9, 17][4]

$$\Pi_d(u,\partial_x) = \sum_{n=0}^{\infty} \frac{1}{n!\,(-u\cdot\partial_u - \frac{d-5}{2})_n} \left(\frac{u^2\,\partial_x^2 - (u\cdot\partial_x)^2}{16}\right)^n . \tag{2.5}$$

Let us consider an infinite set of couplings of ϕ to external higher spin fields h_s through these currents:

$$S_{\text{int}}[\phi,h] = \sum_{s=0}^{\infty} \frac{1}{s!} \int d^d x\, J^{\mu_1\cdots\mu_s}\, h_{\mu_1\cdots\mu_s}\,. \tag{2.6}$$

Introducing

$$h(x,u) = \sum_{s=0}^{\infty} \frac{1}{s!}\, h_{\mu_1\cdots\mu_s}(x)\, u^{\mu_1}\cdots u^{\mu_s}\,, \tag{2.7}$$

the coupling (2.6) may be written also as

$$S_{\text{int}}[\phi,h] = \int d^d x\ h(x,\partial_u)\, J(x,u)\,\big|_{u=0}\,. \tag{2.8}$$

Due to the transversality and tracelessness of the currents on the scalar mass shell, these couplings are invariant under

$$\delta_{\text{lin}}\, h_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\varepsilon_{\mu_2\cdots\mu_s)} + \eta_{(\mu_1\mu_2}\,\alpha_{\mu_3\cdots\mu_s)}\,, \tag{2.9}$$

[4]Here $(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}$ is the Pochhammer symbol.

provided ϕ is subject to its free equations of motion. These are linearised conformal higher spin (CHS) transformations [13]. Off the scalar mass shell, these symmetries are deformed to the nonlinear CHS ones [16, 17] generalising the diffeomorphism and Weyl transformations of the Weyl gravity

$$\delta_{\rm CHS}\, h_{\mu_1\cdots\mu_s} = \delta_{\rm lin}\, h_{\mu_1\cdots\mu_s} + \mathcal{O}(h)\,, \qquad\qquad \delta_{\rm CHS}\, \phi = \mathcal{O}(\phi)\,. \tag{2.10}$$

For $s = 0$ the field h_0 is a scalar coupled to $J_0 = \phi^*\phi$, for $s = 1$ we get a coupling of a vector h_μ to $\Upsilon(1)$ current, and for $s = 2$ we get linearised metric $h_{\mu\nu}$ coupled to energy-momentum tensor.[5] The higher spin couplings are natural generalisations of these lower spin couplings.

Next, we may supplement $S_{\rm free}[\phi] + S_{\rm int}[\phi, h]$ with the dynamical action for CHS fields h_s. The functional of h_s invariant under (2.10) can be identified with the local UV divergent part of the induced action found by integrating out some number of additional scalars [16, 17]. The induced action (discussed already in the Introduction, see (1.3), (1.4), (1.6)) may be written as [17]

$$\Gamma[h] = \int d^d p\, k(p)\, \left(p^2\right)^{\frac{d-4}{2}}\, G(X,Y)\, \tilde h(p,u_1)\, \tilde h(-p,u_2)\Big|_{u_i=0} + \mathcal{O}(h^3)\,, \tag{2.11}$$

where $\tilde h(p,u)$ is the Fourier transform of $h(x,u)$ in (2.7) and $k(p)$ is a spin-independent function

$$k(p) = c_1 \log \frac{p^2}{\Lambda^2} + c_2\ . \tag{2.12}$$

Λ is a UV cutoff (we omit power divergences) and c_1, c_2 are simple numerical constants. The operator $G(X,Y)$ acting on u_1, u_2 is given by

$$G(X,Y) = \sum_{s=0}^{\infty} \frac{\Gamma(\frac{d-3}{2})}{2^{4s}\,\Gamma(s+\frac{d-3}{2})\,\Gamma(s+\frac{d-1}{2})}\, C_s^{(\frac{d-3}{2})}\Big(\frac{X}{\sqrt{Y}}\Big)\, Y^{\frac{s}{2}}\,, \tag{2.13}$$

where $C_s^{(\lambda)}(z)$ is the Gegenbauer polynomial and X and Y are differential operators defined by

$$\begin{aligned} X &= p^2\, \partial_{u_1}\cdot\partial_{u_2} - p\cdot\partial_{u_1}\, p\cdot\partial_{u_2}\,, \\ Y &= \left[(p\cdot\partial_{u_1})^2 - p^2\,\partial_{u_1}^2\right]\left[(p\cdot\partial_{u_2})^2 - p^2\,\partial_{u_2}^2\right]. \end{aligned} \tag{2.14}$$

Keeping only the singular $\log\Lambda$ part of $k(p)$ or, equivalently, replacing it by a renormalized constant $\kappa = c_1 \log\mu^2$ (proportional to the number N of scalars that were integrated out and playing the role of the overall inverse coupling constant) we may define the local CHS action as

$$S_{\rm CHS}[h] = \kappa \int d^d p\, \left(p^2\right)^{\frac{d-4}{2}}\, G(X,Y)\, \tilde h(p,u_1)\, \tilde h(-p,u_2)\Big|_{u_i=0} + \mathcal{O}(h^3)\,. \tag{2.15}$$

[5] Other standard scalar coupling terms such as $h_\mu h^\mu \phi^*\,\phi$ for electrodynamics and $\frac{(d-2)}{4(d-1)} R\,\phi^*\,\phi$ for Weyl gravity can be absorbed into a redefinition of h_0.

The quadratic part of (2.15) represents a collection of free conformal spin s actions [13]

$$S_{\text{CHS},2}[h_s] \sim \kappa \int d^d x\; h_s\, \mathrm{P}_s\, \partial^{2s+d-4}\, h_s\,, \tag{2.16}$$

where as in (1.6) the operator P_s is transverse traceless projector. $S_{\text{CHS},2}[h_s]$ is invariant under (2.9) and in $d=4$ may be interpreted as the square of the linearised spin s analog of Weyl tensor. The important point here is that the relative normalisation of conformal spin s fields in the induced action are fixed by the coupling $S_{\text{int}}[\phi, h]$ (2.6) (other choices of normalisation would break the CHS symmetries (2.10)).[6]

3 Four-scalar tree-level scattering amplitude

Given the system of CHS fields coupled to a free scalar via (2.6), we can study the simplest four-scalar scattering process with the exchange of all CHS fields (figure 1). This provides an interesting example when the issue of definition of the sum over all spins becomes important. Ref. [9] analysed a similar process where the exchanged particles were the standard massless Fronsdal higher spin ones. There, the scattering amplitude was obtained as a function of infinitely many undetermined coupling constants between the massless higher spin fields and a scalar. In the present case all the $\phi - \phi - h_s$ coupling constants are fixed up to an overall factor (the coupling constant κ^{-1} of the CHS theory) and as a result the amplitude will be given by an explicit expression in terms of a sum over spins.

3.1 Conformal spin s exchange

To compute the relevant four-scalar amplitude we start with the vertex (2.6) and consider integrating over h_s (in quadratic approximation only) while keeping ϕ as external fields:

$$\begin{aligned}&\langle S_{\text{int}}[\phi, h]\, S_{\text{int}}[\phi, h]\,\rangle_0 \\ &\qquad = \sum_{s=0}^{\infty} \int \frac{d^d p}{(2\pi)^d}\, \frac{1}{(s!)^2}\, \tilde{J}^{\mu_1\cdots\mu_s}(p) \left\langle \tilde{h}_{\mu_1\cdots\mu_s}(p)\, \tilde{h}_{\nu_1\cdots\nu_s}(-p) \right\rangle_0 \tilde{J}^{\nu_1\cdots\nu_s}(-p)\,. \end{aligned} \tag{3.1}$$

Here $\tilde{J}_s$ are the Fourier transforms of the bilinear conserved currents in (2.2) and the free propagators of the CHS fields are (in transverse traceless gauge)

$$\left\langle \tilde{h}_{\mu_1\cdots\mu_s}(p)\, \tilde{h}^{\nu_1\cdots\nu_s}(-p) \right\rangle_0 = \frac{n_s}{2\kappa\; s!}\, \frac{\mathrm{P}^{\nu_1\cdots\nu_s}_{\mu_1\cdots\mu_s}(p)}{(p^2)^{s+\frac{d-4}{2}}}\,, \tag{3.2}$$

where $\mathrm{P}^{\nu_1\cdots\nu_s}_{\mu_1\cdots\mu_s}(p) = \delta^{\nu_1\cdots\nu_s}_{\mu_1\cdots\mu_s} + \ldots$ is the projector to transverse traceless totally symmetric tensors and κ is the overall coefficient in (2.15). Since the propagators are contracted with traceless and conserved currents (the external scalar legs are assumed to be on-shell), all other terms denoted by dots in P_s will drop out.

[6] One can also compute the h^3 term in the local CHS-invariant $\log\Lambda$ part of the induced action [16, 17]. Extending the construction of the non-linear CHS action to higher orders in h_s appears to be technically non-trivial and may require a new method which is non-perturbative in number of fields (see in this connection discussions of the unfolding program for CHS fields [24–26]).

The coefficients n_s in (3.2) are given by the normalisation of the quadratic part in (2.15). That they are completely fixed is equivalent to the fact that the $\phi-\phi-h_s$ coupling constants are all fixed. Explicitly, eq. (2.15) contains different tensor structures represented by different monomials in X and Y. As we have remarked before, since the propagators are contracted with traceless conserved currents, only traceless and transverse terms are relevant. The Y operator contains at least one trace or divergence, so it is sufficient to consider only the Y-independent part of the CHS action, i.e. to expand $G(X,Y)$ in (2.13) as

$$G(X,Y)=\sum_{s=0}^{\infty}\frac{\Gamma(\frac{d-3}{2})}{2^{3s}\,\Gamma(s+\frac{d-1}{2})}\,\frac{X^s}{s!}+\mathcal{O}(Y)\,. \tag{3.3}$$

As a result, one finds

$$n_s=\frac{2^{3s}\,\Gamma\big(s+\frac{d-1}{2}\big)}{\Gamma(\frac{d-3}{2})}\,. \tag{3.4}$$

Let us represent (3.1) as a sum over spins

$$\langle S_{\rm int}[\phi,h]\,S_{\rm int}[\phi,h]\,\rangle_0=\kappa^{-1}\sum_{s=0}^{\infty}n_s\,V_s\,, \tag{3.5}$$

where the spin s contribution is found to be

$$\begin{aligned} V_s\;&=\;\frac{1}{2\,s!}\int\frac{d^dp}{(2\pi)^d}\;\tilde{J}^{\mu_1\cdots\mu_s}(p)\,\frac{1}{(p^2)^{s+\frac{d-4}{2}}}\;\tilde{J}_{\mu_1\cdots\mu_s}(-p)\\ &=\;\frac{1}{2\,s!}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2)^{s+\frac{d-4}{2}}}\,(\partial_{u_1}\cdot\partial_{u_2})^s\,\Pi_d(u_1,i\,p)\,\tilde{\mathfrak{J}}(p,u_1)\,\tilde{\mathfrak{J}}(-p,u_2)\Big|_{u_i=0}\,, \end{aligned} \tag{3.6}$$

where Π_d was defined in (2.5). The Fourier transform of the traceful-current generating function (2.4) is given by

$$\begin{aligned} \tilde{\mathfrak{J}}(p,u)\;&=\;\int d^dx\,e^{-i\,x\cdot p}\,\phi^*\Big(x+\frac{i}{2}\,u\Big)\,\phi\Big(x-\frac{i}{2}\,u\Big)\\ &=\;\int\frac{d^dk\,d^d\ell}{(2\pi)^{2d}}\;\tilde{\phi}^*(k)\,\tilde{\phi}(\ell)\,e^{u\cdot\frac{k+\ell}{2}}\,(2\pi)^d\,\delta^{(d)}(p+k-\ell)\,. \end{aligned} \tag{3.7}$$

Using this expression we can represent V_s in (3.5) as

$$\begin{aligned} V_s\;=\;\frac{1}{2}\int\frac{d^dk_1\,d^d\ell_1\,d^dk_2\,d^d\ell_2}{(2\pi)^{4d}}\,(2\pi)^d\,\delta^{(d)}(k_1+k_2-\ell_1-\ell_2)\\ \times\,\tilde{\phi}^*(k_1)\,\tilde{\phi}(\ell_1)\,\tilde{\phi}^*(k_2)\,\tilde{\phi}(\ell_2)\,A_s(k_1,k_2,\ell_1,\ell_2)\,, \end{aligned} \tag{3.8}$$

where A_s is the spin-s exchange amplitude $(p=k_1-\ell_1=\ell_2-k_2)$

$$A_s(k_1,k_2,\ell_1,\ell_2)=\frac{1}{2\,(p^2)^{s+\frac{d-4}{2}}}\,\frac{(\partial_{u_1}\cdot\partial_{u_2})^s}{s!}\,\Pi_d(u_1,i\,p)\,e^{\frac{1}{2}[u_1\cdot(k_1+\ell_1)+u_2\cdot(k_2+\ell_2)]}\Big|_{u_i=0}\,. \tag{3.9}$$

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Using the explicit expression for Π_d in (2.5) the resulting t-channel amplitude due to spin s exchange is found to be

$$\begin{aligned} A_s^{(\mathsf{t})}(\mathsf{s},\mathsf{t},\mathsf{u}) &= \frac{1}{2(-4)^s\,(-\mathsf{t})^{\frac{d-4}{2}}}\sum_{n=0}^{[s/2]}\frac{1}{2^{2n}\,n!\,(s-2n)!\,(-s-\frac{d-5}{2})_n}\left(\frac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\right)^{s-2n} \\ &= \frac{1}{2(-8)^s\,(\frac{d-3}{2})_s}\,\frac{1}{(-\mathsf{t})^{\frac{d-4}{2}}}\,C_s^{(\frac{d-3}{2})}\left(\frac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\right). \end{aligned} \tag{3.10}$$

Here $\mathsf{s},\mathsf{t},\mathsf{u}$ are the Mandelstam variables (with $\mathsf{s}+\mathsf{t}+\mathsf{u}=0$ in the present massless scalar case) and $C_n^{(\lambda)}(z)$ is the Gegenbauer polynomial.

Since the theory under consideration is conformal, the amplitude has a manifestly scale-covariant form. In particular, in $d=4$ it depends only on ratio of the Mandelstam variables (also, in $d=4$ the Gegenbauer polynomial reduces to the Legendre one).

The total summed over spins t-channel amplitude is thus given by (cf. (3.5), (3.8))

$$A^{(\mathsf{t})}(\mathsf{s},\mathsf{t},\mathsf{u}) = \kappa^{-1}\sum_{s=0}^{\infty} n_s\,A_s^{(\mathsf{t})}(\mathsf{s},\mathsf{t},\mathsf{u}) = \kappa^{-1}\,\frac{1}{2\,(-\mathsf{t})^{\frac{d-4}{2}}}\,F_d\left(-\,\frac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}\right), \tag{3.11}$$

where the function $F_d(z)$ is given by

$$F_d(z) = \sum_{s=0}^{\infty}\frac{n_s}{2^{3s}\,(\frac{d-3}{2})_s}\,C_s^{(\frac{d-3}{2})}(z)\,. \tag{3.12}$$

Using the expression for n_s in (3.4) , $F_d(z)$ simplifies to

$$F_d(z) = \sum_{s=0}^{\infty}(s+\alpha_d)\,C_s^{(\alpha_d)}(z)\,, \qquad \alpha_d \equiv \frac{d-3}{2}\,. \tag{3.13}$$

For generic values of z , the sum over spins diverges and thus needs to be defined with a certain regularisation prescription.

3.2 Summing over spins

In general, a particular definition of the sum over spins and thus the resulting expressions for the scattering amplitudes should be consistent with the underlying symmetries of the theory.[7] We shall return to this point below but let us first proceed formally, choosing a natural cutoff prescription to define the sum over s. Let us introduce a parameter $w=e^{-\varepsilon}<1$ (with $\varepsilon\to 0$), compute the sum and then define (3.13) as a limit $w\to 1$

$$F_d(z) = \lim_{w\to 1} F_d(z,w)\,, \qquad F_d(z,w) = \sum_{s=0}^{\infty}(s+\alpha_d)\;w^s\,C_s^{(\alpha_d)}(z)\,. \tag{3.14}$$

[7] One may draw an analogy with the Veneziano amplitude in string theory where one also sums over an infinite number of different (massive) field contributions. When computing it in string field theory context, one would also need to choose a particular summation over modes prescription. This prescription is selected automatically in the first-quantised world sheet approach in which the 2d conformal invariance and the associated space-time symmetries are built in.

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We may write $F_d(z,w)$ as

$$F_d(z,w) = w^{1-\alpha_d}\,\frac{d}{dw}\left(w^{\alpha_d}\sum_{s=0}^{\infty} w^s\, C_s^{(\alpha_d)}(z)\right), \tag{3.15}$$

and use the expression for the generating function $\sum_{s=0}^{\infty} w^s\, C_s^{(\alpha_d)}(z) = (1-2zw+w^2)^{-\alpha_d}$ for the Gegenbauer polynomials to define the regularized expression for $F_d(z,w)$ by an analytic continuation:[8]

$$F_d^{\rm reg}(z,w) = \alpha_d\,\frac{1-w^2}{(1-2z\,w+w^2)^{\alpha_d+1}}\,. \tag{3.16}$$

Notice that $F_d^{\rm reg}(z,1)$ happens to vanish for $z\neq 1$, while for $z=1$, we get

$$F_d^{\rm reg}(1,w) = \alpha_d\,\frac{1+w}{(1-w)^{d-2}}\,, \tag{3.17}$$

which diverges as $w\to 1$. Thus $F_d^{\rm reg}(z)$ is a particular distribution with support localised at $z=1$. In fact, it is just proportional to the $(d-4)$-th derivative of the delta-function, i.e.[9]

$$F_d^{\rm reg}(z) = \frac{(-1)^{d-4}}{(d-4)!}\,\delta^{[d-4]}(z-1)\,, \qquad \text{i.e.} \qquad F_4^{\rm reg}(z) = \delta(z-1)\,. \tag{3.18}$$

The above regularisation of the sum over spins is essentially the same as the one used in [1, 5, 7] in the context of higher spin partition functions. In the case of CHS theory in d dimensions (or d-dimensional boundary theory) the sum $\sum_{s=0}^{\infty} f_d(s)$ was first replaced by the convergent sum $\sum_{s=0}^{\infty} e^{-\varepsilon(s+\alpha_d)} f_d(s)$ where $\alpha_d = \frac{d-3}{2}$ and then taking the limit $\varepsilon\to 0$ all $\frac{1}{\varepsilon^n}$ poles were dropped.

The same result (3.18) is found also using another natural regularisation prescription utilizing integral representation for the Gegenbauer polynomials. For simplicity, let us focus on the $d=4$ case where (3.13) reduces to

$$F_4(z) = \sum_{s=0}^{\infty}\left(s+\frac{1}{2}\right)\,P_s(z)\,. \tag{3.19}$$

Here $P_s = C_s^{(1/2)}$ is the Legendre polynomial. The idea is to use the integral representation

$$P_s(z) = \frac{1}{\pi}\int_0^{\pi} dx\,\left(z+\sqrt{z^2-1}\,\cos x\right)^s\,, \tag{3.20}$$

and interchange the summation over s with the integration. Performing first the sum we find the following integrand

$$\sum_{s=0}^{\infty}\left(s+\frac{1}{2}\right)\left(z+\sqrt{z^2-1}\,\cos x\right)^s = \frac{z+1+\sqrt{z^2-1}\,\cos x}{2\,(z-1+\sqrt{z^2-1}\,\cos x)^2}\,. \tag{3.21}$$

[8] The radius of convergence of the series in w is not greater than 1 (it is 1 when $|z|<1$ and e^{-x} when $|z|=\cosh x\geq 1$) so the direct evaluation of $F_d(z,1)$ gives a divergent expression.

[9] Starting with (3.16) and changing the variables $z=x+w$, $\epsilon^2 = 1-w^2$ we get $F_d^{\rm reg}(x,\epsilon) = \alpha_d\,\frac{\epsilon^2}{(x^2+\epsilon^2)^{\alpha_d+1}}$. As a result, $F_d^{\rm reg}(z) = \lim_{\epsilon\to 0} F_d^{\rm reg}(x,\epsilon) = \frac{(-1)^{d-4}}{(d-4)!}\,\delta^{[d-4]}(x)\,.$

Here we have also used an analytic continuation since for any $x \in [0, \pi]$, there exists such z that the series is divergent. Performing the x-integral we get

$$F_4^{\text{reg}}(z) = \frac{1}{\pi} \int_0^\pi dx \, \frac{z + 1 + \sqrt{z^2 - 1} \, \cos x}{2\,(z - 1 + \sqrt{z^2 - 1} \, \cos x)^2} = \delta(z - 1)\,, \tag{3.22}$$

i.e. the same result as in (3.18).

3.3 Total amplitude in $d = 4$

In the case of a complex scalar scattering $\phi\,\phi \to \phi\,\phi$ in $d = 4$ one finds the total amplitude by adding the t-channel and the u-channel contributions following from (3.11) and (3.18), (3.22)

$$A_{\phi\phi\to\phi\phi} = \frac{\kappa^{-1}}{4} \left[\delta\left(\frac{\mathsf{s}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{s}}{\mathsf{u}}\right) \right] \,. \tag{3.23}$$

This unfamiliarly looking amplitude actually vanishes due to massless kinematics. Indeed, choosing the c.o.m. frame ($\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$) and introducing the scattering angle θ for which $\cos\theta = \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1||\vec{p}_3|}$ one can show (using $E_i = |\vec{p}_i|$) that $\frac{\mathsf{s}}{\mathsf{t}} = -\frac{1}{\sin^2\frac{\theta}{2}}$, $\frac{\mathsf{s}}{\mathsf{u}} = -\frac{1}{\cos^2\frac{\theta}{2}}$.[10] Thus the arguments of the delta-functions never vanish for real θ, i.e. we get

$$A_{\phi\phi\to\phi\phi} = 0\,. \tag{3.24}$$

For the $\phi\,\phi^* \to \phi\,\phi^*$ scattering, we find

$$A_{\phi\phi^*\to\phi\phi^*} = \frac{\kappa^{-1}}{4} \left[\delta\left(\frac{\mathsf{u}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{u}}{\mathsf{s}}\right) \right] = \frac{\kappa^{-1}}{4} \left[\delta\left(\cot^2\frac{\theta}{2}\right) - \delta\left(\cos^2\frac{\theta}{2}\right) \right], \tag{3.25}$$

where the two delta-functions correspond to the t-channel and the s-channel contributions, respectively. These two contributions cancel each other, so that again the total amplitude vanishes

$$A_{\phi\phi^*\to\phi\phi^*} = 0\,. \tag{3.26}$$

One may also consider the real scalar case when only the even spin currents in (2.2) are non-vanishing and thus only the even spin CHS exchanges are contributing. Then only the even z part of the function in (3.12), (3.22) is relevant and we get for the total amplitude

$$\begin{aligned} A^{(\mathbb{R})}_{\phi\phi\to\phi\phi} &= \frac{\kappa^{-1}}{8} \left[\delta\left(\frac{\mathsf{u}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{s}}{\mathsf{t}}\right) + \delta\left(\frac{\mathsf{u}}{\mathsf{s}}\right) + \delta\left(\frac{\mathsf{t}}{\mathsf{s}}\right) + \delta\left(\frac{\mathsf{t}}{\mathsf{u}}\right) + \delta\left(\frac{\mathsf{s}}{\mathsf{u}}\right) \right] \\ &= \frac{\kappa^{-1}}{8} \left[\delta\left(\cot^2\frac{\theta}{2}\right) - \delta\left(\csc^2\frac{\theta}{2}\right) - \delta\left(\cos^2\frac{\theta}{2}\right) \right. \\ &\qquad \left. - \delta\left(\sin^2\frac{\theta}{2}\right) + \delta\left(\tan^2\frac{\theta}{2}\right) - \delta\left(\sec^2\frac{\theta}{2}\right) \right]. \end{aligned} \tag{3.27}$$

[10] In general, there may be a possible subtlety in the collinear limit when $p_1^\mu = r p_2^\mu$ and one cannot go to the c.o.m. frame but this limit requires complex momenta and its significance in the present context is unclear.

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Here the first two delta-functions come from the t-channel, the middle two from the s-channel and the last two from the u-channel exchange. The contributions from the three channels cancel against each other and we get again

$$A^{(\mathbb{R})}_{\phi\phi\to\phi\phi} = 0\,. \tag{3.28}$$

Thus, while the individual spin s exchange contributions are nontrivial, the total amplitude vanishes if computed with a particular prescription for summation over spins. As we shall argue below, the vanishing of the four scalar scattering amplitude is actually implied by the global CHS symmetry of the theory.

4 Constraints of conformal higher spin symmetry on scalar amplitudes

We have seen that the tree-level scattering amplitude vanishes when a particular regularisation is used to define the summation over all exchanged spins. The principle that should be selecting one regularization over the other should be the preservation of underlying symmetries of the theory.[11]

The system of CHS fields coupled to massless scalar has the *global* CHS symmetry which plays an analogous role to Lorentz or conformal symmetry in standard field theory. One may thus require the consistency of a prescription of summation over spins with this symmetry. For example, the introduction of the regularization factor w^s in (3.14) may be implemented by adding it to the CHS propagator in (3.2). This translates into the following modification of the quadratic part of the CHS action (2.15) (see (2.13), (2.14))

$$S^{\rm reg}_{\rm CHS,2}[h;w] = \int d^dp\, \left(p^2\right)^{\frac{d-4}{2}}\, G(w^{-1}X, w^{-2}Y)\,\tilde h(p,u_1)\,\tilde h(-p,u_2)\Big|_{u_i=0}\,. \tag{4.1}$$

One may then ask if this regularized action still preserves the global CHS symmetry which is reviewed in appendix A.

Below we will demonstrate that the vanishing of the tree amplitude found in the previous section is actually implied by the invariance under a particular subset of global CHS symmetry transformations. This provides an evidence of a consistency of the regularization of the sum over spins used in section 3.

Assuming that CHS symmetry is free from anomalies,[12] we would like to analyze how the global CHS symmetry of the scalar action coupled to the CHS fields constrains the correlators (and thus the scattering amplitudes) of massless scalar fields. The global

[11] One possible analogy is with summation over the Kaluza-Klein modes in a 5d theory compactified on a circle. Viewed as a 4d theory it involves sum over an infinite number of KK mode contributions with manifest symmetry being only 4d Lorentz symmetry, but the requirement of preservation of the original 5d Lorentz symmetry should impose constraints on how one should perform the sum to recover the result found directly in 5d.

[12] Possible anomalies from loop graphs may cancel if one sums over all CHS fields. Indeed, it was demonstrated in [2, 4] that a-coefficient of Weyl anomaly of the $d = 4$ CHS theory vanishes assuming a particular prescription of summation over spins. The same may apply also to the c-coefficient of 4d Weyl anomaly [1, 4, 5, 8]. As the Weyl symmetry is one of the CHS gauge symmetries, this is an indication that the same may apply to all algebraic CHS symmetries.

CHS symmetry should constrain possible interaction terms in the effective action for the scalars (with CHS fields integrated out, i.e. appearing only on internal lines). In fact, it may prohibit any non-trivial interaction terms, i.e. may imply the vanishing of the corresponding S-matrix.

Among the infinitely many global CHS transformations (A.7), let us consider the "hyper-translations" (cf. (A.12)):[13]

$$\delta\phi(x) = \varepsilon^{\mu_1\cdots\mu_r}\,\partial_{\mu_1}\cdots\partial_{\mu_r}\,\phi(x)\,. \tag{4.2}$$

Here $\varepsilon^{\mu_1\cdots\mu_r}$ is a constant parameter. For simplicity, let us restrict the discussion to the case of real scalars, so that r will take only odd values. Choosing $\varepsilon^{\mu_1\cdots\mu_r}$ proportional to a product $y^{\mu_1}\dots y^{\mu_r}$ where y^μ is an arbitrary vector we conclude that (4.2) implies also the invariance under

$$\delta\phi(x) = (e^{y\cdot\partial_x} - e^{-y\cdot\partial_x})\phi(x) = \phi(x+y) - \phi(x-y)\,. \tag{4.3}$$

The invariance of the scalar four-point correlation function under such symmetry implies

$$\begin{aligned}&\langle\phi(x_1+y)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle + \langle\phi(x_1)\,\phi(x_2+y)\,\phi(x_3)\,\phi(x_4)\rangle\\ &\quad+\langle\phi(x_1)\,\phi(x_2)\,\phi(x_3+y)\,\phi(x_4)\rangle + \langle\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4+y)\rangle\\ &\quad-(y\leftrightarrow -y) = 0\,.\end{aligned} \tag{4.4}$$

Translated to the momentum space this constraint becomes

$$\sin(p_{12}\cdot y)\,\sin(p_{13}\cdot y)\,\sin(p_{14}\cdot y)\,\langle\tilde\phi(p_1)\,\tilde\phi(p_2)\,\tilde\phi(p_3)\,\tilde\phi(p_4)\rangle = 0\,, \tag{4.5}$$

where $p_{ij} = \frac{1}{2}(p_i+p_j)$ and we have used trigonometric identities and momentum conservation, $p_1+p_2+p_3+p_4=0$. Making special choice of the vector y^μ as

$$y^\mu = a\,p_{12}^\mu + b\,p_{13}^\mu + c\,p_{14}^\mu\,, \tag{4.6}$$

where a,b,c are some arbitrary parameters, and applying the condition (4.5) to the case of the on-shell scattering amplitude of four real scalars (cf. (3.27)) we get (using that $p_i^2=0$)

$$\sin\left(\frac{1}{4}a\,\mathsf{s}\right)\,\sin\left(\frac{1}{4}b\,\mathsf{t}\right)\,\sin\left(\frac{1}{4}c\,\mathsf{u}\right)\,A^{(\mathbb{R})}_{\phi\phi\to\phi\phi}(\mathsf{s},\mathsf{t},\mathsf{u}) = 0\,. \tag{4.7}$$

Since a,b,c are arbitrary, eq.(4.7) is equivalent to $\mathsf{s}\,\mathsf{t}\,\mathsf{u}\,A^{(\mathbb{R})}_{\phi\phi\to\phi\phi}=0$, and its solution is given by the distribution,

$$A^{(\mathbb{R})}_{\phi\phi\to\phi\phi}(\mathsf{s},\mathsf{t},\mathsf{u}) = k_1(\mathsf{t},\mathsf{u})\,\delta(\mathsf{s}) + k_2(\mathsf{u},\mathsf{s})\,\delta(\mathsf{t}) + k_3(\mathsf{s},\mathsf{t})\,\delta(\mathsf{u})\,, \tag{4.8}$$

with arbitrary functions k_i. In addition, we may use also the conformal symmetry which is also a sub-algebra of the CHS symmetry. In particular, in $d=4$ the amplitude should be

[13] Here we shall ignore the trace parts: the trace parts of (4.2) correspond to the trivial symmetries (vanishing on equations of motion) that will not give any useful conditions for the correlators. There is no problem in including such symmetries back if needed.

invariant under the dilatation symmetry (cf. (3.27)), i.e. under the rescaling of momenta by a real constant λ

$$A^{(\mathbb{R})}_{\phi\phi\to\phi\phi}(\lambda^2\,\mathsf{s},\lambda^2\,\mathsf{t},\lambda^2\,\mathsf{u}) = A^{(\mathbb{R})}_{\phi\phi\to\phi\phi}(\mathsf{s},\mathsf{t},\mathsf{u})\,. \tag{4.9}$$

This condition restricts k_i to homogeneous functions of degree one. Moreover, the crossing symmetry of the amplitude requires $k_i(x,y) = k(x+y)$. Combining all these results, we obtain

$$A^{(\mathbb{R})}_{\phi\phi\to\phi\phi}(\mathsf{s},\mathsf{t},\mathsf{u}) = k\left[(\mathsf{t}+\mathsf{u})\,\delta(\mathsf{s}) + (\mathsf{u}+\mathsf{s})\;\delta(\mathsf{t}) + (\mathsf{s}+\mathsf{t})\,\delta(\mathsf{u})\right], \tag{4.10}$$

where k is an overall constant. Finally, imposing the condition $\mathsf{s}+\mathsf{t}+\mathsf{u}=0$, we conclude that the amplitude completely vanishes.

This formal argument appears to apply not only at the tree but also at the loop level if the global CHS symmetry is not anomalous. It should also apply to the complex scalar scattering case. As we have already seen in section 3, the tree-level scalar amplitude indeed vanishes in a particular regularization of the sum under spins which should thus be consistent with the CHS symmetry.

It would be interesting to directly verify this vanishing also for the full one-loop on-shell scalar amplitude. We shall address the computation of the loop amplitude in the next section.

5 One-loop corrections

Let us now turn attention to the quantum corrections. Here we will not compute the full one-loop correction to four-scalar amplitude (which is expected to vanish in view of the symmetry argument in the previous section) but address only the question about UV singular part of the amplitude. We shall consider the case of dimension $d = 4$.

In 4d scalar QED, the four-scalar one-loop amplitude contains logarithmic UV divergence coming from loop diagrams with spin-one propagators, and similar divergences are expected in each conformal higher spin loop. One may ask if these divergences may go away after one sums over all spins, i.e. if four-scalar one-loop S-matrix is UV finite in the model of massless scalar coupled to CHS theory. Below we shall address this question by explicitly calculating such UV divergence.

Since the only coupling constant κ in this theory (2.1), (2.6), (2.15) is dimensionless on dimensional grounds the only possible logarithmic UV divergence in the on-shell effective action is proportional to the local term $\int d^4x\,(\phi^*\phi)^2$. In order to compute the coefficient of this term in the one-loop effective action it is sufficient to consider the background field ϕ to be constant, i.e. to assume that the external legs in four-scalar one-loop amplitude are taken at zero momentum (which is a particular on-shell point in a massless scalar theory, so the result should be gauge-independent). Henceforth we shall focus only on the amplitudes with vanishing external momenta.

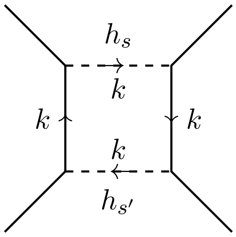

Figure 3. Box diagram with vanishing external momenta.

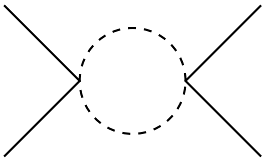

Figure 4. Bubble diagram in scalar QED.

5.1 Diagrams contributing to four-scalar scattering amplitude

Box diagram. Let us first consider the box diagram in figure 3 which involves two scalar propagators and two CHS propagators. Let us recall that the CHS propagator (3.2) is proportional to the transverse-traceless (TT) projector $\mathrm{P}^{\mu_1\cdots\mu_s}_{\nu_1\cdots\nu_s}(k)$ satisfying

$$k_{\mu_1}\,\mathrm{P}^{\mu_1\cdots\mu_s}_{\nu_1\cdots\nu_s}(k) = 0 = \eta_{\mu_1\mu_2}\,\mathrm{P}^{\mu_1\cdots\mu_s}_{\nu_1\cdots\nu_s}(k)\,. \tag{5.1}$$

When all external momenta vanish, the only non-vanishing momentum in the box diagram is the internal momentum k, and $\mathrm{P}_s(k)$ of spin s CHS propagator will be necessarily contracted with k making the diagram vanish. Therefore, the only non-vanishing contribution to the local counterterm $(\phi^*\phi)^2$ may come only from the diagram with $s=s'=0\,$, i.e. from the contribution of the "non-propagating" spin 0 member of the CHS tower (with free action $\int d^4x(h_0)^2$, cf. (2.16)), and is given by

$$A^{(1)}_{\rm Box} = \left(\frac{n_0}{\kappa}\right)^2 I(\Lambda)\,. \tag{5.2}$$

Here n_s is given in (3.4) and $I(\Lambda)$ is the standard UV divergent loop integral,

$$I(\Lambda) = \int^{\Lambda} \frac{d^4k}{(k^2)^2}\,. \tag{5.3}$$

CHS bubble diagram. The fact that the box diagrams with spin $s\geq 1$ exchanges do not give any contribution to UV divergence is similar to the scalar QED case where the UV divergence arises only from the bubble diagram (figure 4) with two $A^\mu A_\mu\,\phi^*\phi$ vertices. In the present case of the scalar coupled to CHS theory, we do not have higher order contact scalar interactions $\mathcal{O}(h^2,\phi^2)$ in the action (1.7). Hence, one might think that no one-loop bubble diagrams can induce $(\phi^*\phi)^2$ term in the effective action because none of them are 1-PI. However, the usual distinction between 1-PI and non-1-PI diagrams does not formally apply in $d=4$ CHS theory due to the presence of non-propagating $s=0$ field which has

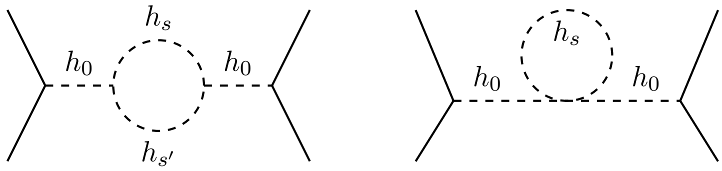

Figure 5. Diagrams contributing to $(\phi^* \phi)^2$.

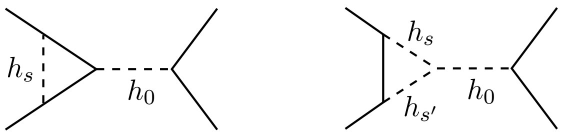

Figure 6. Charge renormalization diagrams.

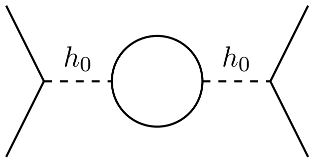

Figure 7. Non 1-PI diagram with scalar loop.

the $(h_0)^2$ kinetic term (see (2.16)). It turns out that the diagrams in figure 5 (where the h_0 lines there are effectively shrunk to a point and h_s loops include also the contributions of the corresponding ghosts) do produce zero-momentum $(\phi^* \phi)^2$ terms. We shall return to the analysis of these contributions in section 5.3.

Charge renormalisation diagrams. The "charge renormalization" diagrams involving the one-loop correction to the $h_s \, \phi^* \, \phi$ vertices may also contribute to the $(\phi^* \, \phi)^2$ contact term through the h_0 internal line (see figure 6).

As in the case of the box diagram, here again the only non-trivial diagrams with constant external scalars are the ones which involve only $s, s' = 0$ internal lines. Moreover, it follows from dimensional analysis that the there is no ${h_0}^3$ vertex in the CHS action so that the only non-vanishing contribution comes from the first diagram in figure 6 with $s = 0$. Its contribution is given by

$$A^{(1)}_{\text{charge ren.}} = \left(\frac{n_0}{\kappa}\right)^2 I(\Lambda) \,. \tag{5.4}$$

Scalar bubble diagram. Finally, there is also a possible contact $(\phi^* \phi)^2$ contribution from the non 1-PI diagram with scalar loop and non-propagating h_0 field in figure 7. We find

$$A^{(1)}_{\text{scalar bubble}} = N_\phi \left(\frac{n_0}{\kappa}\right)^2 I(\Lambda) \,, \tag{5.5}$$

where for generality we included the factor N_ϕ of the number of massless scalars (in the discussion above we had $N_\phi = 1$).

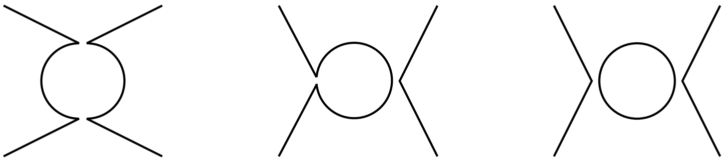

Figure 8. One-loop diagrams with $(\phi^*\phi)^2$ vertices (broken or open lines indicate the origin of these diagrams in relation to diagrams in figures 3, 6, 7).

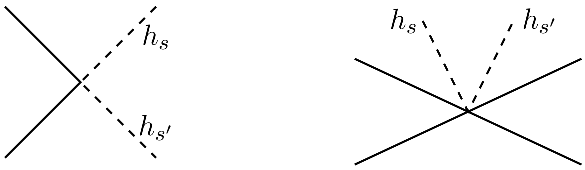

Figure 9. Higher order contact vertices.

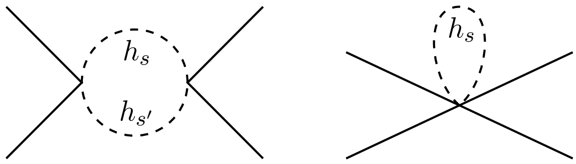

Figure 10. One-loop diagrams with $h^2\,\phi^2$ and $h^2\,\phi^4$ vertices.

5.2 Equivalent approach: integrating out h_0 first

The fact that h_0 is non-propagating allows one to treat it as an auxiliary field, i.e. integrate it out ending up with a local action for the remaining fields ϕ and $h_{s\geq 1}$. The price is getting new interaction vertices.

- First, the CHS action itself will be modified. Since the h_0 equation is of the form $h_0 = \mathcal{O}(h^2_{s\geq 1})$, we get additional vertices at quartic or higher orders. These will not, however, contribute to the four-scalar scattering at the one-loop order.
- The presence of $\phi^*\phi\, h_0$ coupling in (2.6) implies that, after solving for h_0, the massless scalar scalar action acquires the self interaction vertex $(\phi^*\phi)^2$. As a result, there will be extra diagrams in figure 8 contributing to the four scalar scattering. These are, of course, equivalent to the $s = s' = 0$ diagram in figure 3, the first diagram with $s = 0$ in figure 6 and the diagram in figure 7 with all h_0 lines shrunk to a point.
- Finally, there will appear additional interaction vertices between ϕ and $h_{s\geq 1}$, notably, the vertices of type $h^2\,\phi^2$ and $h^2\,\phi^4$ (see figure 9). These lead to extra one-loop diagrams in figure 10, which again are equivalent to the diagrams in figure 5 with h_0 lines shrunk to a point.

In this approach, with h_0 integrated out first, all UV divergences of the four-scalar scattering amplitudes come from two types of 1-PI diagrams: the one (figure 8) involving ϕ-loops and the other one (figure 10) involving h_s-loops (where in general one is also to add ghost loop contributions):

$$A^{(1)}_{\rm tot} = A^{(1)}_{\phi-{\rm loop}} + A^{(1)}_{h_s-{\rm loop}}\,. \tag{5.6}$$

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The former contributions were already given in (5.2), (5.4) and (5.5) and thus we find, symbolically,

$$A^{(1)}_{\phi-\text{loop}} = (1 + 1 + N_\phi) \left(\frac{n_0}{\kappa}\right)^2 I(\Lambda) \,. \tag{5.7}$$

The dependence on N_ϕ makes it clear that $A^{(1)}_{\phi-\text{loop}}$ cannot be canceled by $A^{(1)}_{h_s-\text{loop}}$ since the latter is independent of N_ϕ .

Thus for generic value of N_ϕ the one-loop four scalar scattering amplitude will have a UV divergence, i.e. the amplitude will not vanish contrary to what happened at the tree level. This may not be in contradiction with the CHS global symmetry argument of section 4 because scalar loop contributions may render the CHS symmetry anomalous.

One possible approach is to treat the scalar ϕ field as an external only, i.e. to ignore all diagrams with ϕ scalar loops altogether. It is then of interest to see if the contributions the remaining diagrams with CHS loops only in figure 10 may vanish when summed over all spins. This will be addressed in the next subsection.

5.3 Divergent part of one-loop CHS effective action in constant h_0 background

Let us now consider the diagrams in figure 5 (or equivalently those in figure 10) where the external scalar field ϕ lines are taken at zero momentum (so that same applies to h_0 lines in figure 5). The UV divergent contribution from the diagrams in figure 5 takes the form

$$c_{\text{CHS}} \left(\frac{n_0}{\kappa}\right)^2 I(\Lambda) \int d^4x \, (\phi^* \, \phi)^2 \,, \tag{5.8}$$

where the coefficient c_{CHS} encodes the contributions from infinitely many CHS field loops (figure 10). Equivalently, this constant appears in the UV divergent h_0 dependent part of the one-loop effective action of the CHS theory

$$\Gamma^{(1)}_{\text{div}}[\, h_0] = c_{\text{CHS}} \, I(\Lambda) \int d^4x \, (h_0)^2 \,. \tag{5.9}$$

On general grounds, the CHS theory $S_{\text{CHS}} = \kappa \int d^4x(h_0^2 + F^2_{\mu\nu} + C^2_{\mu\nu\lambda\rho} + \ldots)$ having dimensionless coupling constant should be renormalizable (the gauge symmetries fix the local action uniquely) and thus the same $c_{\text{CHS}} \log \Lambda$ one-loop coefficient should appear in front of the (linearised) Weyl tensor term if spin 2 background is turned on in addition to h_0 in (5.9). Then c_{CHS} should be the same as the conformal anomaly c-coefficient of the CHS theory. The conformal anomaly a-coefficient of the CHS theory (corresponding to topological Euler number divergence in the effective action) was found in [2, 4, 5] to vanish if a natural regularization for summation over all spins is used. The same vanishing was found also for the total c-coefficient [1, 4, 8] under the assumption that contributions to conformal anomaly from higher derivative CHS operators on Ricci flat background factorize. One may thus expect that total c_{CHS} coefficient of the UV divergent h_0^2 term in (5.9) should also vanish.

To check this let us directly evaluate the logarithmically divergent part of the one-loop effective action of CHS theory assuming that the only non-trivial background is the constant

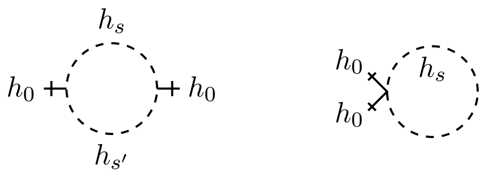


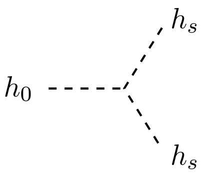
Figure 11. CHS effective action in h_0 background.

spin 0 field h_0 . To compute $c_{\rm CHS}$ from the diagrams of figure 10 we need to take into account both the "physical" (gauge-fixed) field loop and the ghost loop contributions, i.e.

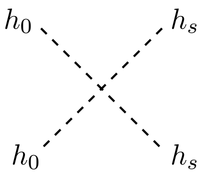
$$c_{\rm CHS} = c^{ph}_{\rm CHS} + c^{gh}_{\rm CHS}\,. \tag{5.10}$$

5.3.1 Physical field loop contribution

Let us first consider the loop diagrams involving physical fields. There are two types of 1-PI diagrams in figure 11 and their evaluation requires the knowledge of $h_0\, h_s\, h_{s'}$ and ${h_0}^2\, {h_s}^2$ vertices. These vertices can be represented in momentum space as



$$= \kappa\, \tilde h_0(0)\, C_s(k,\partial_{u_1},\partial_{u_2})\, \tilde h_s(k,u_1)\, \tilde h_s(-k,u_2)\,\big|_{u_i=0}\,, \tag{5.11}$$



$$= \kappa\, (\tilde{h_0(0)})^2\, Q_s(k,\partial_{u_1},\partial_{u_2})\, \tilde h_s(k,u_1)\, \tilde h_s(-k,u_2)\,\big|_{u_i=0}\,. \tag{5.12}$$

Here the two functions C_s and Q_s encode all tensor structures:

$$\begin{aligned} C_s(k,\partial_{u_1},\partial_{u_2}) &= c_s\, (k^2)^{s-1}\, (\partial_{u_1}\cdot\partial_{u_2})^s + \dots\,, \\ Q_s(k,\partial_{u_1},\partial_{u_2}) &= q_s\, (k^2)^{s-2}\, (\partial_{u_1}\cdot\partial_{u_2})^s + \dots\,. \end{aligned} \tag{5.13}$$

Here dots stand for terms involving at least one trace or one divergence of a field so that they drop out in the traceless and transverse gauge that we shall assume. For the same reason we can consider only $h_0\,(h_s)^2$ vertices instead of more general $h_0\, h_s\, h_{s'}$ ones because the latter necessarily contain a trace or divergence.

Using the vertices (5.13) we get, respectively, for the left and the right diagram in figure 11

$$\begin{aligned} I_1 &= \frac{1}{4}\left(\frac{n_s}{\kappa}\right)^2 \int d^4k\, \frac{\kappa\, C_s(k,\partial_{u_1},\partial_{u_2})\, \kappa\, C_s(k,\partial_{v_1},\partial_{v_2})\, \mathrm{P}_s(k,u_1,v_1)\, \mathrm{P}_s(k,u_2,v_2)}{(k^2)^{2s}}\bigg|_{u_i=v_i=0}\,, \\ I_2 &= \frac{1}{4}\,\frac{n_s}{\kappa} \int d^4k\, \frac{\kappa\, Q_s(k,\partial_{u_1},\partial_{u_2})\, \mathrm{P}_s(k,u_1,u_2)}{(k^2)^s}\bigg|_{u_i=0}\,, \end{aligned} \tag{5.14}$$

where we have used the propagator (3.2) involving the traceless and transverse projector P_s. After removing the auxiliary variables u_i and v_i (which amounts to the contraction of all the indices) and performing the k-integral (which reduces to the UV divergent term (5.3)), we obtain

$$I_1 = \frac{1}{4}(2s+1)\,(n_s\,c_s)^2\,I(\Lambda)\,, \qquad\qquad I_2 = \frac{1}{4}\,(2s+1)\,n_s\,q_s\,I(\Lambda)\,. \tag{5.15}$$

Here the factor $2s+1$ comes from the trace of the projector P_s (this is the dimension of the symmetric rank s representation of $so(3)$ which is the traceless and transverse part of 4d Lorentz tensor). The cubic (5.11) and quartic interactions (5.12), or equivalently the coefficients $n_s\,c_s$ and $n_s\,q_s$ in (5.15) can be extracted from the CHS action. This is done in appendix B and with the result being

$$n_s\,c_s = -4\left(s+\frac{1}{2}\right)\,, \quad s \geq 1\,; \qquad n_s\,q_s = 8\left(s+\frac{1}{2}\right)\left(s-\frac{1}{2}\right)\,, \quad s\geq 2\,. \tag{5.16}$$

Finally, using the above expressions, we obtain

$$c^{ph}_{\mathrm{CHS}} = 2^3\sum_{s=1}^{\infty}\left(s+\frac{1}{2}\right)^3 + 2^2\sum_{s=2}^{\infty}\left(s+\frac{1}{2}\right)^2\left(s-\frac{1}{2}\right)\,. \tag{5.17}$$

The sum over spins is formally divergent and thus requires an appropriate definition or regularization to be discussed below.

5.3.2 Ghost loop contribution

To find the ghost contribution corresponding to the traceless transverse gauge let us consider the gauge symmetries of the classical CHS action. Since we are interested in computing the one-loop ghost contribution in a constant h_0 background, it is sufficient to consider the classical CHS action to quadratic order in all $s>0$ fields, i.e. with h_0-dependent kinetic operator

$$S_{\mathrm{CHS}} = \int d^4x\,\langle h|K(h_0)|h\rangle\,, \tag{5.18}$$

where $\langle\cdot|\cdot\rangle$ stands for the contraction of indices. When the background h_0 is turned off, the operator K reduces to that of the free CHS theory. The above action is invariant under the following gauge transformation (cf. appendix A)

$$\delta_{\epsilon,\alpha}\,h = u\cdot\partial_x\,\epsilon + \left[u^2 - h_0\,\mathcal{F}(\partial_u,\partial_x)\right]\alpha\,, \tag{5.19}$$

where the gauge fields and parameters can be chosen to be doubly-traceless and traceless, respectively, without loss of generality. The h_0 dependent part of gauge transformation is given with the operator $\mathcal{F}(\partial_u,\partial_x) = \Pi_d(\partial_u,\partial_x)\,\Pi^{-1}_{d+4}(\partial_u,\partial_x)\,.$ In the following, we shall gauge fix the CHS field h to traceless and transverse one by making use of the transformation (5.19).

First, using the α part of the transformation (5.19), we can gauge fix the trace of h to zero. This step does not introduce any ghost (since the transformation is algebraic) but modifies the residual gauge transformation to the form

$$\delta_\epsilon\,h = T(h_0,\epsilon) = \mathrm{P}_{\mathrm{T}}\left[u\cdot\partial_x - h_0\,\mathcal{G}(\partial_u,\partial_x)\right]\epsilon\,, \tag{5.20}$$

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where $\mathrm{P_T}$ is the traceless projector and the precise form of $\mathcal{G}(\partial_u, \partial_x)$ is given in appendix C. Due to the tracelessness of the parameter ϵ the gauge transformation (5.20) remains linear in h_0 even after this traceless gauge fixing.

Second, using the remaining transformation (5.20), we can make the traceless field h also transverse. This step involves differential part of the gauge transformation which gives rise to a non-trivial Jacobian. The latter can be represented by an appropriate ghost contribution (see appendix C), with the ghost action being

$$S_{\rm gh} = \int d^4x \, \langle \bar{c} | K_{\rm gh}(h_0) | c \rangle \,, \tag{5.21}$$

$$K_{\rm gh}(h_0) = \partial_x \cdot \partial_u \frac{\delta T(h_0, \epsilon)}{\delta \epsilon} = \partial_x \cdot \partial_u \, \mathrm{P}_T \, [u \cdot \partial - h_0 \, \mathcal{G}(\partial_u, \partial_x)] \,. \tag{5.22}$$

The crucial observation is that by shifting appropriately the ghost fields c one can completely eliminate all h_0 dependence. This is to be done spin by spin, starting with the lower spin, so that each ghost field is shifted once and then left alone. It is important to note that the existence of this redefinition is due to an additional divergence term in the operator of gauge transformation. The gauge transformation itself cannot be redefined in such a way that it becomes independent of h_0. The details of this argument are given in appendix C.

We thus conclude that the CHS ghosts do not couple to a constant h_0 background, and hence

$$c^{gh}_{\rm CHS} = 0 \,. \tag{5.23}$$

5.3.3 Summing over spins

The final expression for the coefficient of the divergent h_0^2 term in the CHS action may thus be written as (see (5.9), (5.10), (5.17), (5.23))

$$c_{\rm CHS} = -5 + 4 \sum_{s=0}^{\infty} \left[3 \left(s + \frac{1}{2} \right)^3 - \left(s + \frac{1}{2} \right)^2 \right] \,. \tag{5.24}$$

As was mentioned above, this coefficient should be expected to be proportional to the conformal anomaly c-coefficient of the CHS theory. The expression for the latter was found to vanish [4, 5, 8] provided the sum over spins is defined using the $e^{-(s+\alpha)\,\epsilon}$ cutoff with $\alpha = \frac{1}{2}$ just as in the similar vanishing of the a-coefficient [2, 4].[14] The same cutoff factor $e^{-(s+\alpha_d)\epsilon}$ appeared in (3.15) with $\alpha_d = \frac{d-3}{2} = \frac{1}{2}$ in $d = 4$. Using such exponential cutoff $e^{-(s+\alpha)\,\epsilon}$ in (5.24) and dropping all singular terms we get

$$c_{\rm CHS}(\alpha) = \frac{1}{30} \left(90\,\alpha^4 - 140\,\alpha^3 + 75\,\alpha^2 - 15\,\alpha - 152 \right) , \tag{5.25}$$

which does not, however, vanish for a rational value of α.

[14] The computation of the one-loop conformal anomaly c-coefficient in the CHS theory is based on two assumptions: (i) the CHS action obtained as an induced action in near-flat space expansion can be reformulated (using a field redefinition) in such a way that at least quadratic kinetic terms in generic curved metric background are reparametrization and Weyl invariant; (ii) the higher derivative kinetic operators $D^{2s} + \ldots$ — while not factorizing, in general, into products of $D^2 + \ldots$ operators in a Ricci-flat background [27] (as they do in AdS or sphere background) — still contribute to c-anomaly in the same way as if they do factorize (the terms with derivatives of the curvature tensor that obstruct factorization cannot contribute to $C^2_{\mu\nu\kappa\lambda}$ conformal anomaly on dimensional grounds).

The meaning of this observation is unclear at the moment. One possibility is that (5.24) is missing some contributions making it different from the conformal anomaly c-coefficient discussed in [1, 4, 8]. Indeed, the CHS spin s field conformal anomaly coefficients are 6-th order polynomials[15] while the summand in (5.24) is only cubic polynomial in s. At the same time, the partial spin s contributions to h_0^2 and $C^2_{\mu\nu\kappa\lambda}$ divergent terms in the CHS action need not match: what is expected to be the same is only the total summed over spins coefficients.

Another possibility is that while c_{CHS} in (5.24) is not related to the conformal anomaly c-coefficient its still determines the UV divergent part of the CHS loop contribution to the four-scalar amplitude. In that case to resolve the regularization ambiguity it would be important (as in the tree-level amplitude case in (3.14)) to keep the external momentum non-zero (which would play, e.g., the role of z in (3.19)).[16] Equivalently, it would be desirable to repeat the calculation of the CHS effective action in a *non-constant* h_0 background.

6 Concluding remarks

The $d = 4$ conformal higher spin theory having vanishing total coefficients of the conformal anomaly (and thus possibly of all higher symmetry anomalies) is a potentially consistent quantum theory of an infinite tower of higher spin fields having a large amount of symmetry. While apparently non-unitary due to higher derivatives in the $s > 1$ kinetic terms this theory has a well-defined formulation in flat space background and thus deserves a detailed investigation.

Here we have studied the scattering amplitudes for a massless conformal scalar ϕ coupled to CHS theory. The four-scalar tree-level amplitude is given by the exchange of the whole tower of CHS fields. We have found that under a natural prescription of summation over spins the resulting tree level amplitude vanishes. This vanishing turns out to be in agreement with the expectation based on global extended conformal symmetry.

We also addressed the extension of this computation to one-loop order. We considered only the simplest case of vanishing external scalar momenta. The one-loop diagrams contributing to the four-scalar scattering are of two types: (i) involving internal scalar propagators (i.e. scalar loops), and (ii) involving only CHS field loops. The former are potentially anomalous (scalar loop in external CHS background has, in particular, a non-vanishing Weyl anomaly) and thus the symmetry argument of section 3 about the vanishing of the total amplitude due to global CHS symmetry need not apply. We have thus concentrated on the CHS loop contributions only. The expectation is that the coefficient c_{CHS} of the UV divergent term in the zero-momentum amplitude (or of the $(\phi^*\phi)^2$ term in the effective action) should be the same as the conformal anomaly c-coefficient and should

[15] Explicitly, $\mathrm{a}_s = \frac{1}{720}\nu_s(3\nu_s + 14\nu_s^2)$ [2, 4] and $\mathrm{c}_s - \mathrm{a}_s = \frac{1}{720}\nu_s(4 - 45\nu_s + 15\nu_s^2)$ where $\nu_s = s(s+1)$.

[16] In the case of the one-loop partition function of the massless higher spin theory in AdS it was noticed [5] that a consistent result can be obtained by first summing over spins and then removing the UV cut-off. In our case as well, first summing over the spins and then sending momentum p to zero may lead to more sensible result than first setting momentum to zero and then summing over spins.

thus vanish after summation over all spins. The expression for $c_{\rm CHS}$ we have found does not vanish however and that issue requires further investigation. It would be important, in particular, to clarify the precise relation between the coefficient $c_{\rm CHS}$ found in section 5 and the conformal anomaly c-coefficient.[17]

It would be interesting also to apply the methods of the present paper to the computation of the tree and one-loop S-matrix for the CHS fields themselves (e.g., Maxwell vector and Weyl graviton). That may provide further evidence for the existence of a consistent regularization of the sum over spins and may also shed some light on the (non)unitarity issue.

Acknowledgments

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A Review of global CHS symmetry

Let us review the origin of the global CHS symmetry and its action on the free scalar and CHS fields following [16, 17].

Transformation of massless scalar field. The massless scalar action (2.1) may be written in the following *operator* representation

$$S_{\rm free}[\phi] = \langle\phi|\, \hat{p}^2\, |\phi\rangle\,, \tag{A.1}$$

where $\phi(x) = \langle x|\phi\rangle$ and $\hat{p}_\mu = i\,\partial_\mu$. To find the maximal symmetries of this action we consider the most general transformation linear in ϕ. In the operator formulation, it reads

$$\delta\,|\phi\rangle = i\,\hat{t}\,|\phi\rangle\,, \tag{A.2}$$

where $\hat{t}$ is an arbitrary polynomial in $\hat{x}$ and $\hat{p}$, i.e. a differential operator acting on $\phi(x)$. The condition that it preserves the action (A.1) is

$$\hat{p}^2\,\hat{t} = \hat{t}^\dagger\,\hat{p}^2\,. \tag{A.3}$$

[17]To recall, the main logical steps were as follows. The coefficient $c_{\rm CHS}$ we computed was the coefficient of the $h_0^2 \log\Lambda$ term in the CHS effective action. The conformal anomaly c-coefficient is the same as the coefficient of the $C^2_{\mu\nu\lambda\rho}\log\Lambda$ term in the CHS effective action. The full UV divergent term in the one-loop effective action should be invariant not only under the Weyl symmetry and reparametrizations but also under the whole CHS gauge symmetry. As there is a unique local functional which is invariant under CHS gauge symmetry [16, 17], the coefficients of the h_0^2 and $C^2_{\mu\nu\lambda\rho}$ terms are thus expected to be the same.

This defines the maximal symmetries of conformal scalar action up to the *trivial* ones

$$\delta\phi^i = C^{ij}(\phi)\,\frac{\delta S_{\rm free}}{\delta\phi^j}\,, \qquad\qquad C^{ij} = -C^{ji}\,, \tag{A.4}$$

which are proportional to the equations of motions, i.e. vanish on-shell. Such trivial transformations correspond in the case of (A.1) to the operator of the form

$$\hat t = \hat r\,\hat p^2\,, \qquad\qquad \hat r^\dagger = \hat r\,, \tag{A.5}$$

with $\hat r$ an arbitrary hermitian factor. The set of operators $\hat t$ satisfying (A.3) modulo (A.5) defines the global CHS symmetry that acts on conformal scalars as in (A.2).[18]

A convenient way to treat the operators is by using the Wigner-Weyl correspondence (see, e.g., appendix A in [17]). Then we can map the operator $\hat t$ to a phase-space function

$$t(x,p) = e(x,p) + i\,a(x,p) \equiv (e,a)\,, \tag{A.6}$$

and all operator products become Moyal products. In this formulation the conformal scalar transforms as

$$\delta\phi(x) = e^{-\frac{i}{2}\partial_{x_2}\cdot\partial_u}\,t\,(x_1,u)\,\phi(x_2)\,\Big|_{\substack{x_1=x_2=x\\ u=0}}\,, \tag{A.7}$$

where the conditions on $\hat t$ in (A.6) to represent the CHS symmetry are

$$p\cdot\partial_x\,e - (p^2+\partial_x^2)a = 0\,, \tag{A.8}$$

$$(e,a) \sim (e,a) + \big((p^2+\partial_x^2)\,r\,,\,p\cdot\partial_x\,r\big)\,. \tag{A.9}$$

The algebraic structure is induced from the operator product as (here the commutators in the r.h.s. are defined using the Moyal $\star$ product)

$$\Big[\,(e_1\,,\,a_1)\,,(e_2\,,\,a_2)\Big] = \big(\,[\,e_1\overset{\star}{,}e_2\,] - [\,a_1\overset{\star}{,}a_2\,]\,,\,[\,e_1\overset{\star}{,}a_2\,] + [\,a_1\overset{\star}{,}e_2\,]\,\big)\,. \tag{A.10}$$

The global CHS symmetry contains the conformal algebra with generators

$$P_\mu = (p_\mu,0)\,, \qquad M_{\mu\nu} = (x_{[\mu}\,p_{\nu]},0)\,, \qquad K_\mu = (x_\mu\,x\cdot p, x_\mu)\,, \qquad D = (x\cdot p,1)\,. \tag{A.11}$$

and also other higher spin generators, for example, the generators of *hyper-translations*

$$P_{\mu_1\ldots\mu_r} = (p_{\{\mu_1}\ldots p_{\mu_r\}},\ 0)\,, \tag{A.12}$$

where $\{\ldots\}$ indicates the subtraction of all traces.

[18] In fact, the global CHS symmetry in d dimensions is nothing but Vasiliev's HS algebra in $(d+1)$ dimensions. The typical formulation of Vasiliev's HS algebra involves differential operators in $(d+2)$-dimensions, while here we formulated it in terms of differential operators in d-dimensions. The reason for the existence of the two descriptions is the fact that the conformal scalar in d-dimension can be formulated in $(d+2)$-dimensions where the role of $\hat p^2$ is played by the three operators $\hat X^2$, $2(\hat X\cdot\hat P+\hat P\cdot\hat X)$, $\hat P^2$, which form an $\mathfrak{sp}(2,\mathbb{R})$ algebra. See [28] for a recent overview of the HS algebra.

Transformation of CHS fields. The above symmetry may be also considered as a global part of the gauge symmetry acting on the CHS fields. It will then be a symmetry of the action of the free scalars coupled to the CHS fields as in (2.6).

The action of this conformal higher spin symmetry on the CHS fields becomes more transparent in the so-called dressed formulation [16], where one uses a different set of CHS fields $\mathfrak{h}(x,u)$ which related to the original one (2.7) by

$$\mathfrak{h}(x,u) = \Pi_d(\partial_u,\partial_x)\, h(x,u)\,, \tag{A.13}$$

where Π_d was defined in (2.5) (see [17] for details). The CHS action (2.15) then becomes a non-diagonal functional

$$S_{\text{CHS}}[\mathfrak{h}] = \int d^dx\, U_{\frac{d-3}{2}}\Big((\partial_{x_{12}}\cdot\partial_{u_{12}})^2 - \partial_{x_{12}}^2\,\partial_{u_{12}}^2\Big)\,\mathfrak{h}(x_1,u_1)\,\mathfrak{h}(x_2,u_2)\Big|_{\substack{x_1=x_2=x\\ u_1=u_2=0}} + \mathcal{O}(\mathfrak{h}^3)\,, \tag{A.14}$$

where $U_\nu(z) = (\sqrt{z}/2)^{-\nu}\,J_\nu(\sqrt{z}/2)$ (J_ν is a Bessel function). The advantage of working with $\mathfrak{h}$ is that the CHS gauge symmetry takes a simple form

$$\delta\,\mathfrak{h} = [\,\mathfrak{e}\overset{\star}{,}\,u^2+\mathfrak{h}\,] + \{\,\mathfrak{a}\overset{\star}{,}\,u^2+\mathfrak{h}\,\} = \delta^{(0)}\mathfrak{h} + \delta^{(1)}\mathfrak{h}\,, \tag{A.15}$$

where $\star$ acts on the space of functions in x and u. $\delta^{(0)}$ and $\delta^{(1)}$ are respectively $\mathfrak{h}$-independent and $\mathfrak{h}$-linear parts, and the gauge parameters are related to those in (2.9) by

$$\begin{aligned}
\mathfrak{e}(x,u) &= \Pi_{d+2}(\partial_u,\partial_x)\;\epsilon(x,u) + (\partial_x\cdot\partial_u)\;\Pi_{d+2}(\partial_u,\partial_x)\;\frac{1}{2(d-1)+4\,u\cdot\partial_u}\,\alpha(x,u)\,,\\
\mathfrak{a}(x,u) &= \Pi_{d+4}(\partial_u,\partial_x)\;\alpha(x,u)\,.
\end{aligned} \tag{A.16}$$

The field-independent part of the transformation reads

$$\delta^{(0)}\mathfrak{h} = u\cdot\partial_x\,\mathfrak{e} + \left(u^2 - \frac{1}{4}\,\partial_x^2\right)\mathfrak{a}\,. \tag{A.17}$$

This coincides with the l.h.s. of (A.8) and the equivalence relation (A.9) can be interpreted here as a "gauge for gauge" symmetry,

$$\delta\mathfrak{e} = \left(u^2 - \frac{1}{4}\,\partial_x^2\right) r\,, \qquad \delta\mathfrak{a} = -u\cdot\partial_x\, r\,. \tag{A.18}$$

Hence for the special parameter $(\mathfrak{e},\mathfrak{a}) = (e,a)$ satisfying $\delta^{(0)}\mathfrak{h} = 0$ (which can be interpreted as the conformal Killing equation (A.8)) the CHS action (A.14) is invariant under

$$\delta\,\mathfrak{h} = [\,e\overset{\star}{,}\,\mathfrak{h}\,] + \{\,a\overset{\star}{,}\,\mathfrak{h}\,\}\,. \tag{A.19}$$

This defines the action of the global CHS symmetry on the CHS fields. Since it acts linearly, it preserves all different $\mathfrak{h}^n$-parts of the CHS action separately; in particular, it leaves its quadratic part in (A.14) invariant.

The interaction (2.6) between the CHS fields $\mathfrak{h}$ and the conformal scalar (with currents written in the *un-dressed* form, cf. (2.3), (2.8))

$$S_{\text{int}}[\phi,h] = \int d^dx\,\mathfrak{h}(x,\partial_u)\,\mathfrak{J}(x,u)\,\big|_{u=0}\,, \tag{A.20}$$

is also invariant under the global CHS symmetry. This becomes manifest by writing it in the operator form as

$$S_{\text{int}}[\phi, h] = \langle\phi|\, \hat{\mathfrak{h}}\, |\phi\rangle, \tag{A.21}$$

where $\hat{\mathfrak{h}}$ is the operator corresponding to the symbol $\mathfrak{h}(x,p)$.

B Cubic and quartic vertices in the CHS action involving constant h_0 field

Let us start with recalling that given the heat kernel expansion for the massless scalar kinetic operator in conformal higher spin background,

$$\mathrm{Tr}\left[e^{-t\left(\hat{p}^2+\hat{\mathfrak{h}}\right)}\right] = \sum_{n=0}^{\infty} t^{n-2}\, a_n[\mathfrak{h}]\,, \tag{B.1}$$

the local CHS action in $d=4$ can be defined as the second Seeley coefficient (i.e. as the coefficient of the logarithmic UV divergence in the induced action)

$$S_{\text{CHS}}[\mathfrak{h}] \propto a_2[\mathfrak{h}]\,. \tag{B.2}$$

Let us separate the spin-0 part of CHS field $\mathfrak{h}_0$ from the rest of the fields $\mathfrak{h}'$:

$$\mathfrak{h}(x,u) = \mathfrak{h}_0(x) + \mathfrak{h}'(x,u)\,. \tag{B.3}$$

Here $\mathfrak{h}(x,u)$ is defined in (2.7), (A.13) (the distinction between $\mathfrak{h}(x,u)$ and $h(x,u)$ will not be important in traceless transverse gauge). Then restricting $\mathfrak{h}_0$ to be constant one obtains

$$a_n[\mathfrak{h}] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, (\mathfrak{h}_0)^m\, a_{n-m}[\mathfrak{h}']\,. \tag{B.4}$$

In particular,

$$a_2[\mathfrak{h}] = a_2[\mathfrak{h}'] - \mathfrak{h}_0\, a_1[\mathfrak{h}'] + \frac{1}{2}\, (\mathfrak{h}_0)^2\, a_0[\mathfrak{h}'] + \mathcal{O}(\mathfrak{h}_0^3)\,. \tag{B.5}$$

The heat kernel coefficients a_n were calculated in [17] up to quadratic order in $\mathfrak{h}$,

$$\begin{aligned} a_{2+m}[\mathfrak{h}] \;=\; & \int \frac{d^4x}{(4\pi)^2} \sqrt{\frac{\pi}{8}} \left(\frac{1}{2}\partial_{x_{12}}^2\right)^m U_{m+\frac{1}{2}}\left((\partial_{x_{12}}\cdot\partial_{u_{12}})^2 - \partial_{x_{12}}^2\partial_{u_{12}}^2\right) \\ & \qquad\qquad \times \mathfrak{h}(x_1,u_1)\, \mathfrak{h}(x_2,u_2)\Big|_{\substack{x_1=x_2=x\\ u_1=u_2=0}} + \mathcal{O}(\mathfrak{h}^3)\,, \end{aligned} \tag{B.6}$$

$$\begin{aligned} a_{1-m}[\mathfrak{h}] \;=\; & \int \frac{d^4x}{(4\pi)^2} \Bigg[\delta_{m,1} + \left(\frac{1}{4}\,\partial_u^2\right)^m \mathfrak{h}(x,u)\Big|_{u=0} \\ & \qquad + \sqrt{\frac{\pi}{8}}\, V_m(\partial_{x_{12}}, \partial_{u_{12}})\, \mathfrak{h}(x_1,u_1)\, \mathfrak{h}(x_2,u_2)\Big|_{\substack{x_1=x_2=x\\ u_1=u_2=0}} + \mathcal{O}(\mathfrak{h}^3)\Bigg]\,, \end{aligned} \tag{B.7}$$

where

$$V_m(\partial_x, \partial_u) = \left(\frac{1}{4}\,\partial_u^2\right)^{m+1} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{8}\,\partial_x^2\,\partial_u^2\right)^k}{\Gamma(k+m+2)}\, U_{k+\frac{1}{2}}\left((\partial_x\cdot\partial_u)^2\right)\,, \tag{B.8}$$

and $U_\nu(z)$ is the same as in (A.14), i.e.

$$U_\nu(z) = \left(\frac{\sqrt{z}}{2}\right)^{-\nu} J_\nu\left(\frac{\sqrt{z}}{2}\right) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+m+1)\,2^\nu}\left(-\frac{z}{16}\right)^m\,. \tag{B.9}$$

As a result, the CHS Lagrangian depending on constant $\mathfrak{h}_0$ and traceless and transverse $\mathfrak{h}'$ and written in momentum space reads ($\mathfrak{h}(x) \to \tilde{\mathfrak{h}}(p)$)

$$\begin{aligned}\tilde{\mathcal{L}}_{\text{CHS}}[\mathfrak{h}] \propto \sum_{s=0}^{\infty}&\left[1 - \frac{4}{p^2}\left(s+\frac{1}{2}\right)\tilde{\mathfrak{h}}_0(0) + \frac{8}{p^4}\left(s+\frac{1}{2}\right)\left(s-\frac{1}{2}\right)\left(\tilde{\mathfrak{h}}_0(0)\right)^2 + \mathcal{O}\left(\tilde{\mathfrak{h}}_0^3\right)\right]\\ &\times \frac{\left(p^2\right)^s\,\tilde{\mathfrak{h}}_s(p,\partial_u)\,\tilde{\mathfrak{h}}_s(-p,u)}{2^{3s}\,\Gamma(s+\frac{3}{2})} + \mathcal{O}\left(\tilde{\mathfrak{h}}'^3\right)\,,\end{aligned} \tag{B.10}$$

where $\tilde{\mathfrak{h}}_s(p,u) = \frac{1}{s!}\tilde{\mathfrak{h}}_{\mu_1\ldots\mu_s}(p)u^{\mu_1}\ldots u^{\mu_s}$. Here the non-local terms with negative powers of p^2 should be discarded. Hence the cubic $\mathfrak{h}_0\,\mathfrak{h}_s^2$ terms start from $s=1$ where as the quartic $\mathfrak{h}_0^2\,\mathfrak{h}_s^2$ terms start from $s=2$.

C Gauge fixing and ghost action

In this appendix we shall discuss the ghost action corresponding to the traceless transverse gauge on CHS fields.[19] As we have shown in appendix A, the CHS gauge symmetry takes a more concise form (A.15) in "dressed" basis of fields (defined by (A.13), (2.5), (2.7)). It is thus more convenient to fix the gauge in that basis. After all, in the transverse traceless (TT) gauge we will use, the two bases become equivalent: $\mathfrak{h}(x,u)|_{\text{TT}} = h(x,u)|_{\text{TT}}$. In addition, the scalar parts coincide with each other, $\mathfrak{h}_0 = h_0$, independently of the gauge choice.

Restricting to the case where the only non-trivial background is constant h_0, the symmetry transformation (A.15) reduces to the form,

$$\delta\,\mathfrak{h}(x,u) = u\cdot\partial_x\,\mathfrak{e}(x,u) + \left(u^2 - \frac{1}{4}\,\partial_x^2 + h_0\right)\mathfrak{a}(x,u)\,, \tag{C.1}$$

where the fields $\mathfrak{h}$ are doubly-traceless while the parameters $\mathfrak{e}$ and $\mathfrak{a}$ are traceless. We first gauge fix $\mathfrak{h}$ to be traceless utilizing the algebraic part of the symmetry (C.1) generated by $\mathfrak{a}$. Let us note that this gauge fixing requires in principle a finite transformation rather than an infinitesimal one. In fact, the transformation (C.1) is symmetry of the classical action (5.18) even for finite parameters due to its quadratic nature. Imposing $\partial_u^2(\mathfrak{h}+\delta\,\mathfrak{h}) = 0$, we get the relation between $\mathfrak{a}$ and $\mathfrak{e}$ as

$$\mathfrak{a}(x,u) = -\frac{1}{2(2+u\cdot\partial_u)}\,\partial_x\cdot\partial_u\,\mathfrak{e}(x,u)\,, \tag{C.2}$$

[19] A discussion of an alternative gauge leading to simple gauge-fixed action for free conformal higher spin fields in flat space and the corresponding ghost fields may be found in ref. [29].

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and the traceless CHS fields transform now as $\delta\,\mathfrak{h} = T(h_0, \mathfrak{e})$ with

$$T(\mathfrak{h}_0, \mathfrak{e})(x,u) = \mathrm{P}_{\mathrm{T}}\left(u\cdot\partial_x + \frac{\frac{1}{4}\,\partial_x^2 - h_0}{2(2+u\cdot\partial_u)}\,\partial_u\cdot\partial_x\right)\mathfrak{e}(x,u)\,. \tag{C.3}$$

Here, P_{T} is the traceless projector which is $\mathrm{P}_{\mathrm{T}} = 1 - \frac{u^2(\partial_u)^2}{4(s-2)+2d+u^2(\partial_u)^2}$ when acting on a spin s tensor.

Next, let us further gauge fix the traceless CHS field to make it also transverse by using the transformation (C.3). Following the standard Faddeev-Popov procedure, this step of transverse gauge-fixing introduces the ghost action

$$\begin{aligned} S_{\mathrm{gh}} &= \int d^4x\,\langle\bar{c}\,|\,\partial_x\cdot\partial_u\,\frac{\delta T(h_0,\mathfrak{e})}{\delta\,\mathfrak{e}}\,|\,c\rangle \\ &= \int d^4x\,\sum_{s=0}^{\infty}\langle\bar{c}_s|\,\partial_x\cdot\partial_u\,\mathrm{P}_{\mathrm{T}}\left(u\cdot\partial_x|c_s\rangle + \frac{\frac{1}{4}\,\partial_x^2 - h_0}{2(s+3)}\,\partial_u\cdot\partial_x\,|c_{s+2}\rangle\right)\,. \end{aligned} \tag{C.4}$$

Here $c(x,u) = \sum_{s=0}^{\infty} c_s(x,u)$ with $c_s(x,u) = \frac{1}{s!}\,c_{\mu_1\cdots\mu_s}(x)\,u^{\mu_1}\cdots u^{\mu_s}$ is the generating function for the ghost fields and $\langle a|b\rangle = \frac{1}{s!}\,a_{\mu_1\cdots\mu_s}\,b^{\mu_1\cdots\mu_s}$ is the index contraction. Since the gauge parameter $\mathfrak{e}$ is traceless, the ghost c and antighost $\bar{c}$ are both traceless.

For further analysis, we decompose the ghost c into traceless transverse (TT) components as

$$c_s(x,u) = \mathrm{P}_{\mathrm{T}}\sum_{r=0}^{s}(u\cdot\partial_x)^{s-r}\,c_{s,r}(x,u)\,,\qquad\qquad \partial_u^2\,c_{s,r} = 0 = \partial_x\cdot\partial_u\,c_{s,r}\,. \tag{C.5}$$

By plugging this decomposition for c_s and c_{s+2} into the action (C.4), one can observe that the first two TT components $c_{s+2,s+2}$ and $c_{s+2,s+1}$ of c_{s+2} drop out in the summand. We thus end up with

$$S_{\mathrm{gh}} = \int d^4x\,\sum_{s=0}^{\infty}\sum_{r=0}^{s}\,\langle\bar{c}_s|\,\partial_x\!\cdot\!\partial_u\,\mathrm{P}_{\mathrm{T}}\,(u\!\cdot\!\partial_x)^{s+1-r}\left(|c_{s,r}\rangle + k_{s,r}\left(\frac{1}{4}\,\partial_x^2 - h_0\right)\partial_x^2\,|c_{s+2,r}\rangle\right)\,, \tag{C.6}$$

where

$$k_{s,r} = \frac{(s-r+2)(s+r-3)}{4(s+2)(s+3)}\,. \tag{C.7}$$

As follows from (C.6), one can thus completely remove the h_0 dependence in the ghost action by the ghost field redefinition

$$c'_{s,r} = c_{s,r} + k_{s,r}\big(\frac{1}{4}\,\partial_x^2 - h_0\big)\partial_x^2\,c_{s+2,r}\,. \tag{C.8}$$

For a fixed r, this redefinition acts as a matrix which changes the value of s. Since the form of this matrix is an upper triangular one with the identity diagonal elements, the corresponding Jacobian is simply one. The conclusion is that the ghost determinant contribution is trivial, i.e. does not depend on h_0.

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